Annexure to:

De Villiers, Johann U. and Elze-Mari Roux. 2019. Reframing the retirement saving challenge: Getting to a sustainable lifestyle level. *Journal of Financial Counselling and Planning* 30(2): 277-288.

Derivation of the basic sustainable lifestyle level (SLL) relationship of individuals on a fixed income saving for retirement

Consider an employee on a fixed salary who will be k years old at the end of the current year and plans to retire at the end of the year in which they become z years old. The employee earns a current salary of SAL_k, saves a portion p of this salary for retirement, and uses the remaining portion $(1-p)$ for household consumption. The employee expects their salary to stay constant in real terms for the duration of their employment up to retirement, and plans to continue saving a portion p of this salary towards retirement.

The employee has a balance (RET_k) of funds already saved for retirement. This balance is expected to increase in the subsequent $(z-k)$ years to retirement as a result of additional savings and investment returns. The balance of retirement savings at retirement will then be (RET_z).

The employee expects a real annual rate of return of r on retirement savings. The expected rate of inflation is i per year.

At retirement, we assume that the employee will convert the total retirement savings (RET_z) into a constant lifetime income by purchasing a life annuity indexed to inflation. The life annuity rate (which depends, among other factors, on the age at retirement) is a_z. We define a_z as a real rate. The income that the employee will receive in their first year after retirement (year $z+1$) is therefore $(1+i)a_zRET_z$.

At retirement, the balance of funds saved for retirement (RET_z) will consist of the current savings and the investment income that this amount will have earned, plus the savings in the years leading up to retirement and the investment income on these savings. This is shown in Equation (1) below:

$$RET_z = RET_k[(1+i)(1+r)]^{(z-k)} + \sum_{j=k+1}^{z} p.SAL_k\,[(1+i)]^{(j-k)}[(1+r)(1+i)]^{(z-j)} \qquad \ldots(1)$$

or

$$RET_z = RET_k[(1+i)(1+r)]^{(z-k)} + p.SAL_k.(1+i)^{(z-k)} \sum_{j=k+1}^{z} (1+r)^{(z-j)}$$

Simplifying:

$$RET_z = (1+i)^{(z-k)}\left[RET_k(1+r)^{(z-k)} + p.SAL_k \sum_{j=k+1}^{z} (1+r)^{(z-j)}\right]$$

and:

$$RET_z = (1+i)^{(z-k)}\left[RET_k(1+r)^{(z-k)} + p.SAL_k(1+r)^{z} \sum_{j=k+1}^{z} \left(\frac{1}{1+r}\right)^{(j)}\right] \qquad \ldots(2)$$

But (summing the series):

$$\sum_{j=k+1}^{z} \left(\frac{1}{1+r}\right)^{(j)} = \left(\frac{1+r}{r}\right)\left(\frac{1}{1+r}\right)^{(k+1)}\left[1-\left(\frac{1}{1+r}\right)^{(z-k)}\right]$$

Substituting into (2):

$$RET_z = (1+i)^{(z-k)}\left[RET_k(1+r)^{(z-k)} \right.$$
$$\left. + p.SAL_k(1+r)^{z}\left(\frac{1+r}{r}\right)\left(\frac{1}{1+r}\right)^{(k+1)}\left[1-\left(\frac{1}{1+r}\right)^{(z-k)}\right]\right]$$

And therefore:

$$RET_z = (1+i)^{(z-k)}\left[RET_k(1+r)^{(z-k)} + p.SAL_k.(1+r)^{z-k}\left(\frac{1}{r}\right)\left[1-\left(\frac{1}{1+r}\right)^{(z-k)}\right]\right]$$

Leaving:

$$RET_z = [(1+i)(1+r)]^{(z-k)}\left[RET_k + p.SAL_k.\left(\frac{1}{r}\right)\left[1-\left(\frac{1}{1+r}\right)^{(z-k)}\right]\right]$$

… (3)

Equation (3) describes the amount the employee will save for retirement, and this can be used to calculate the income that the employee will receive in their first year of retirement (year $z+1$) as $(1+i)a_z RET_z$.

In the last year before retirement, the employee will receive a salary SAL_z, where the salary has grown according to the rate of inflation from its current level:

$$SAL_z = SAL_k(1+i)^{(z-k)}$$

The income that the employee has available to spend in the last year before retirement is then $(1-p)$ times this final year of salary. For a sustainable lifestyle level, the income that the employee will receive from their accumulated retirement savings in the first year of retirement has to be the same in real terms as the amount they had available to spend in the last year before they retire. This means that the income the employee receives must be $(1+i)$ times more in nominal terms, or:

$$RI_{z+1} = (1+i)(1-p)SAL_k(1+i)^{(z-k)}$$

$$RI_{z+1} = (1-p)SAL_k(1+i)^{(z-k+1)}$$

…(4)

The income in the first year of retirement (RI_{z+1}) that the employee needs to sustain their lifestyle as it was at its pre-retirement level must be the same as the income that their retirement savings can provide $[(1+i)a_z RET_z]$, and therefore:

$$(1+i)a_z RET_z = RI_{z+1}$$

Substituting for (RET_z) from (3) and for (RI_{z+1}) from (4) then yields:

$$(1+i)a_z[(1+i)(1+r)]^{(z-k)}\left[RET_k + p.SAL_k.\left(\frac{1}{r}\right)\left[1-\left(\frac{1}{1+r}\right)^{(z-k)}\right]\right] = (1-p)SAL_k(1+i)^{(z-k+1)}$$

Dividing by $\left[(1+i)^{(z-k+1)}.SAL_k\right]$ yields:

$$a_z(1+r)^{(z-k)}\left[\frac{RET_k}{SAL_k}+p\left(\frac{1}{r}\right)\left[1-\left(\frac{1}{1+r}\right)^{(z-k)}\right]\right] = (1-p)$$

Solving for *p*:

$$a_z(1+r)^{(z-k)}\frac{RET_k}{SAL_k}+pa_z(1+r)^{(z-k)}\left(\frac{1}{r}\right)\left[1-\left(\frac{1}{1+r}\right)^{(z-k)}\right] = (1-p)$$

$$pa_z\left[\frac{(1+r)^{(z-k)}-1}{r}\right]+p = 1-a_z(1+r)^{(z-k)}\frac{RET_k}{SAL_k}$$

$$p=\frac{1-a_z(1+r)^{(z-k)}\frac{RET_k}{SAL_k}}{1+a_z\left[\frac{(1+r)^{(z-k)}-1}{r}\right]}$$

The SLL is then the proportion of the employee’s salary that they do not have to save towards retirement, or:

$$SLL_k=(1-p)SAL_k$$